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GROWTH AND UNIQUENESS OF RANK

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ABSTRACT
We prove that algebras of sub-exponential growth and, more generally, rings with a sub-exponential "growth structure" have the unique rank property. In the opposite direction the proof shows that if the rank is not unique one gets lower bounds on the exponent of growth. Fixing the growth exponent it shows that an isomorphism between free modules of greatly differing ranks can only be implemented by matrices with entries of logarithmically proportional high degrees.

It is well known that there exist rings over which the rank of a free left module is not uniquely defined, i.e., they have modules that are free on two bases of different cardinalities. A ring for which this distressing phenomenon does not happen is said to have the (left) *unique rank property*; we also say that it has the "UR" property or, simply, that it "has UR". A commutative ring always has the UR property since it has a non-trivial (i.e., with 1 going to 1) homomorphism into a field. More generally a ring has the UR property whenever it has a non-trivial homomorphism into a ring that has the UR property. But many rings to which our theorem (below) applies do not have (apriori) such homomorphisms.

It is easy to see that if a ring R has a subring S which has UR and R is a finitely generated free module over S then R has UR. In this note we generalize this fact in a somewhat unexpected direction. We show that if S satisfies some restrictions (for example S may be Noetherian) and R is generated, over S, in a certain "controlled" way then R still has the UR property. The most general condition on S is the content of the next definition. It generalizes commutativity (which was our assumption in an earlier draft).

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DEFINITION 1. *A ring B is (left) monotone if the following holds: for every imbedding of free left B modules $M \hookrightarrow N$, if $M \cong B^m$ and $N \cong B^n$ then $m \leqq n$.*

Of course a monotone ring has the UR property. Unfortunately, many "natural" rings, e.g., free (non-commutative) rings, are not monotone. On the other hand, it is clear that a ring (or algebra) is monotone if all its finitely generated subrings (or subalgebras) are monotone. It is also easily checked that Artinian rings are monotone: one uses the length function. Since a Noetherian commutative ring has a localization into an Artinian ring we see (using the next lemma!) that commutative rings are monotone.

LEMMA 1. *If the ring B has a non-trivial homomorphism into a ring C such that C is monotone and is flat as a (right) B module then B is monotone.*

The prime example for the lemma is when the map $B \to C$ is Ore type localization. If B is semi-prime Noetherian such a map with C Artinian, exists (this is 'Goldie's theorem').

PROOF. Since C is B-flat the induced map $C^m \cong C \otimes_B M \to C \otimes_B N \cong C^n$ is injective. C being monotone gives $m \leqq n$. □

Concerning the monotoneity of general (non-semi-prime) Noetherian rings the following result shows that rings of a certain class, which includes the Goldie (and, in particular, Noetherian) rings, are monotone.

LEMMA 2. *Let B be a ring such that there exists a real-valued function δ defined on the set of isomorphism classes of finitely generated free left B-modules satisfying the following assumptions:*

(1) *δ is non-negative and not identically zero;*

(2) *δ is monotone, i.e., if M injects into N then $\delta(M) \leqq \delta(N)$;*

(3) *δ is additive, i.e., $\delta(M_1 \oplus M_2) = \delta(M_1) + \delta(M_2)$.*

Then B is monotone.

The proof is obvious.

To use the lemma we observe that the *uniform dimension* defined by Goldie (see [2], Ch. 1) — which exists for free modules over a Goldie ring, and more generally whenever free modules have "finite uniform dimension" — is such a function.

The next definition explicates what we said above about the 'controlled generation of R over S'.

Definition 2. Let R be a ring. A (left) growth structure for R is a (left) monotone subring S and an increasing filtration of R by left S sub-modules $R^{(0)} \subseteq R^{(1)} \subseteq \cdots$ such that the following conditions are satisfied:

(i) $S \subseteq R^{(0)}$ and $\bigcup_{k=0}^{\infty} R^{(k)} = R$.

(ii) Each $R^{(k)}$ is a free left S module of finite rank.

(iii) $R^{(k)} \cdot R^{(l)} \subseteq R^{(k+l)}$ for each $k, l \geqq 0$.

(iv) ("finite generation") There exists a positive integer n_0 such that, for $n > n_0$, $R^{(n)} = R^{(n-n_0)} \cdot R^{(n_0)}$.

If $x \in R^{(n)}$ we say that length$(x) \leqq n$. Its length equals n if it is not in $R^{(n-1)}$. The example to keep in mind with respect to this definition is when R is 'group-graded' over S. This means that there is a finitely generated group G and a basis u_σ ($\sigma \in G$) of R over S (i.e., $R = \Sigma\, Su_\sigma$) such that the u_σ's are invertible, normalize S (i.e., $u_\sigma S \subseteq Su_\sigma$) and multiply as in G (i.e., $u_\sigma u_\tau \in Su_{\sigma\tau}$). Crossed products are good examples and were our motivation in this note, see [1]. If $\{\sigma_1, \ldots, \sigma_k\}$ is a finite generating set of G we can define $R^{(n)}$ to be the left S module generated by the elements $\{u_\sigma : \sigma$ can be expressed as a word of length $\leqq n$ in the letters $\sigma_1^{\pm 1}, \ldots, \sigma_k^{\pm 1}\}$. Then it is clear that this filtration satisfies the conditions of Definition 2.

Given a growth structure of R we define its *growth* to be the function

$$\varphi(n) = \mathrm{rank}_S(R^{(n)}).$$

We now generalize Milnor's remarks in [4] and show

Proposition. *$\varphi(n)$ grows at most exponentially. Moreover, the sequence $(\log \varphi(n))/n$ is convergent.*

For the proof of the proposition we shall need the following technical

Lemma 3. *Let $n \geqq m$ be positive integers and $m \geqq n_0$. If $pm \leqq n < (p+1)m$ and $qn_0 \leqq m < (q+1)n_0$ then $R^{(n)} \subseteq (R^{(m)})^{p+1} \cdot (R^{n_0})^p$.*

Here $(R^{(l)})^k$ means $R^{(l)} \cdots R^{(l)}$ k-times.

Proof of Lemma 3. Clearly $pqn_0 \leqq pm \leqq n$. Iterating condition (iv) of Definition 2 pq times we get $R^{(n)} = R^{(n-pqn_0)} \cdot (R^{(n_0)})^{pq}$. Now,

$$n - pqn_0 = n - p(q+1)n_0 + pn_0 \leqq n - pm + pn_0 \leqq m + pn_0.$$

This inequality implies

$$R^{(n-pqn_0)} \subseteq R^{(m+pn_0)} = R^{(m)} \cdot (R^{(n_0)})^p.$$

Altogether we see that

$$R^{(n)} \subseteq R^{(m)} \cdot (R^{(n_0)})^{pq+p}.$$

Replacing each $(R^{(n_0)})^q$ by the larger $R^{(m)}$ we get the conclusion of the lemma. □

Proof of the Proposition. It is enough to prove the second part. Let n, m, p be as in Lemma 3. By monotoneity of S

$$\varphi(n) \leqq \varphi(m)^{p+1} \cdot \varphi(n_0)^p.$$

This implies

$$\frac{1}{n}\log\varphi(n) \leqq \frac{p+1}{n}\log\varphi(m) + \frac{p}{n}\log\varphi(n_0)$$

$$\leqq \frac{p+1}{pm}\log\varphi(m) + \frac{p}{pm}\log\varphi(n_0).$$

If $n \to \infty$ (and m is kept fixed) then as p is, approximately, n/m it also goes to infinity and we see that

$$\limsup_{n\to\infty}\frac{1}{n}\log\varphi(n) \leqq \frac{1}{m}\log\varphi(m) + \frac{1}{m}\log\varphi(n_0) < \infty.$$

Letting $m \to \infty$ we get

$$\limsup_{n\to\infty}\frac{1}{n}\log\varphi(n) \leqq \liminf_{m\to\infty}\frac{1}{m}\log\varphi(m)$$

which completes the proof. □

In the case that

$$\lim \frac{1}{n}\log\varphi(n) = \lambda > 0$$

we say that the filtration grows exponentially with exponent λ. If

$$\lim \frac{1}{n}\log\varphi(n) = 0$$

the filtration grows "sub-exponentially". We should note that when $\varphi(n)$ grows sub-exponentially, it may happen that even the sequence $\log\varphi(n)/\log n$ remains bounded, in which case the growth is "polynomial" of "degree" lim sup(log $\varphi(n)$/log n). This number — which need not be an integer! — can be thought of as the 'Gelfand–Kirillov' dimension of the given growth structure. Until recently it was not known if groups of sup-exponential but non-polynomial growth exist at all. Then Grigorchuk [3] exhibited many such groups which are, moreover, groups of automorphisms of a certain infinite tree. Thus these groups act naturally on some spaces of functions and the corresponding crossed products (with or without cocycle) provide nice examples of sub-exponential (non-polynomial) growth structures.

We now come to the main result of this note. To explain it we introduce some notation which will serve for the rest of the paper.

Let R be a ring with a growth structure $\{S, R^{(n)}\}$. If M is a free R module and $E = \{e_1, \ldots, e_s\}$, $F = \{f_1, \ldots, f_t\}$ are bases of M over R, we define $(E : F)$ as follows. Write

$$e_k = \sum_{l=1}^{t} b_{k,l} f_l, \qquad k = 1, \ldots, s$$

where $b_{k,l} \in R$. Then

$$(E : F) = \max_{k,l} \ \mathrm{length}(b_{k,l}).$$

THEOREM (*with the above notation*). *Let λ be the growth exponent of R, then* $\log(s/t) \leqq \lambda \cdot (E : F)$. *In particular, if the growth is sub-exponential* ($\lambda = 0$) *then $s \leqq t$, i.e., R has the* UR *property* (*by symmetry*); *in fact in this case R is even monotone.*

(The last part of the theorem was observed by L. Rowen.)

PROOF. Let $M^{(n)} = R^{(n)}e_1 + \cdots + R^{(n)}e_s$ and $L^{(n)} = R^{(n)}f_1 + \cdots + R^{(n)}f_t$. It is clear that all $M^{(n)}$ and $L^{(n)}$ are free S modules with

$$\mathrm{rank}_S(M^{(n)}) = s\varphi(n), \qquad \mathrm{rank}_S(L^{(n)}) = t\varphi(n).$$

Let $(E : F) = c$. Since each $b_{k,l} \in R^{(c)}$ we see that, for every n, $M^{(n)} \subseteq L^{(n+c)}$. This implies ($S$ is monotone)

$$s\varphi(n) \leqq t\varphi(n + c).$$

Write $\mu = s/t$. Iterating the last inequality r times gives

$$\varphi(n + rc) \geqq \mu^r \cdot \varphi(n).$$

This implies

$$\frac{\log \varphi(n+rc)}{n+rc} \geqq \frac{r \log \mu}{n+rc} + \frac{\log \varphi(n)}{n+rc}.$$

The limit, when $r \to \infty$, gives the desired inequality. The proof of the last statement (Rowen's part) is exactly the same as the above proof and is left to the reader. □

If R does not have the UR property, suppose, in the notation above, that $t < s$. Then the theorem says $s \leqq t \cdot e^{\lambda c}$ with $c = (E:F)$. This can be rephrased as: given a basis $\{f_1, \ldots, f_t\}$ and fixing c there is a bound on the size of the basis which can be expressed in terms of the f_i's with coefficients of length $\leqq c$. Since a free module that has two bases of unequal cardinalities has also bases of arbitrarily large cardinalities the last observation acquires some significance. Fixing t and thinking of s as increasing to infinity it says that $(E:F)$ will also increase to infinity, at least as fast as $(1/\lambda) \cdot \log(s/t)$.

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